THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A regular meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, October 31, 1903. About fifty persons attended the two sessions, including the following thirty-nine members of the Society:

Professor Joseph Bowden, Professor E. W. Brown, Professor F. N. Cole, Miss E. B. Cowley, Miss L. D. Cummings, Dr. W. S. Dennett, Dr. L. P. Eisenhart, Professor Achsah M. Ely, Dr. William Findlay, Professor T. S. Fiske, Dr. A. S. Gale, Dr. G. B. Germann, Dr. C. N. Haskins, Mr. L. L. Jackson, Professor Harold Jacoby, Mr. S. A. Joffe, Dr. Edward Kasner, Dr. O. D. Kellogg, Professor C. J. Keyser, Dr. G. H. Ling, Mr. L. L. Locke, Professor James Maclay, Dr. C. M. Mason, Professor W. H. Metzler, Professor W. F. Osgood, Professor E. D. Roe, Miss I. M. Schottenfels, Professor Charlotte A. Scott, Dr. Arthur Schultze, Mr. Burke Smith, Professor D. E. Smith, Mr. E. R. Smith, Professor P. F. Smith, Professor Virgil Snyder, Professor H. D. Thompson, Professor E. B. Van Vleck, Miss Mary Underhill, Mr. H. E. Webb, Professor R. S. Woodward.

The President of the Society, Professor Thomas S. Fiske, occupied the chair, being relieved by Professor E. W. Brown at the afternoon session. The Council announced the election of the following persons to membership in the Society: Miss G. C. Alden, Westfield, Mass.; Mr. L. D. Ames, University of Missouri; Professor R. C. Archibald, Ladies' College, Sackville, N. B.; Mr. W. H. Bates, Purdue University; Miss H. D. Buckingham, Lexington, Mass.; Miss L. D. Cummings, Vassar College; Mr. Harry English, Washington, D. C.; Professor G. A. Gibson, Glasgow and West of Scotland Technical College; Miss M. F. Gould, Everett, Mass.; Dr. O. D. Kellogg, Princeton University; Mr. W. A. Manning, Stanford University; Dr. C. M. Mason, Massachusetts Institute of Technology; Professor Helen A. Merrill, Wellesley College; Mr. E. A. Miller, Massachusetts Institute of Technology; Mr. E. H. Taylor, Eastern Illinois State Normal School; Professor Anna L. Van Benschoten, Wells College; Mr. R. E. Wilson,

Northwestern University. Nine applications for membership in the Society were received.

A list of nominations for officers and other members of the Council was adopted and ordered placed on the official ballot for the annual election. The office of assistant secretary of the Society was revived and filled by the appointment of Dr. William Findlay. A committee was appointed to make arrangements for holding the next summer meeting of the Society at St. Louis, and to coöperate with the committee of the St. Louis Exposition in organizing the Mathematical Section of the International Congresses. A committee consisting of Professors Joseph Bowden and Gustave Legras was appointed to audit the accounts of the treasurer.

The following papers were presented at this meeting:

(1) Dr. A. S. GALE: "On three types of surfaces of the third order regarded as double surfaces of translation."

(2) Dr. L. P. EISENHART: "Surfaces of Bonnet and their transformations."

(3) Dr. EDWARD KASNER: "On partial geodesic representation."

(4) Professor F. N. COLE: "On the factoring of large numbers."

(5) Professor E. GOURSAT: "A simple proof of a theorem in the calculus of variations" (extract from a letter to Professor W. F. Osgood).

(6) Mr. BURKE SMITH: "On the deformation of surfaces whose parametric lines form a conjugate system."

(7) Professor G. A. MILLER: "On the number of sets of conjugate subgroups."

(8) Mr. ELIJAH SWIFT: "On the condition that a point transformation of the plane be a projective transformation."

(9) Miss I. M. SCHOTTENFELS: "On the simple groups of order $8!/2$" (preliminary communication).

(10) Miss I. M. SCHOTTENFELS: "The necessary condition that two linear homogeneous differential equations shall have common integrals."

Professor Goursat's paper was communicated to the Society through Professor Osgood, Mr. Swift's paper through Professor Bôcher. In the absence of the authors, the papers of Professor Goursat and Mr. Swift were read by Professor Osgood, and Professor Miller's paper was read by title. Professor Cole's paper appeared in the December number of the BULLE-

TIN. Dr. Gale's paper appears in the present number. Abstracts the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

2. In seeking to determine the surfaces admitting a deformation with preservation of the lines of curvature Bonnet found that they are characterized by the property of having the same spherical representation of their lines of curvature as pseudospherical surfaces. Dr. Eisenhart considers these surfaces of Bonnet without any reference to their rôle in the theory of deformation and discovers certain transformations of these surfaces into surfaces of the same kind. When a configuration upon the sphere of the kind considered is given, the determination of all the surfaces of Bonnet with this representation of the lines of curvature requires the integration of an equation of Laplace; all the corresponding surfaces are said to form a group. The moulded surfaces and all the parallels of pseudospherical surfaces are surfaces of Bonnet. A determination is made of all surfaces of Bonnet with plane lines of curvature in one or two systems, and it is shown that the transformation of Lie for pseudospherical surfaces can be generalized so that, when the spherical representation of one group of surfaces of Bonnet is known, the representation of an infinity of other groups can be found directly.

Bianchi has shown that a pseudospherical surface can be transformed into an infinity of surfaces of the same kind whose tangent planes are perpendicular to the tangent plane of the given surface, and that the complete determination of this transformation requires the solution of a Riccati equation. It can be shown that this is only a special case of a similar transformation admitted by all surfaces of Bonnet, and the analytic work is the same as in the particular case. Each of these transforms belongs to a different group, but the determination of all the members of these groups requires the integration of one equation of Laplace. From each point M of a given surface S there diverge an infinity of planes tangent to the different transforms; the points of contact of these planes with the latter lie on the circumference of a circle in a plane parallel to the tangent plane to S and with the normal at M for axis; this gives a geometric interpretation of the cyclic systems associated with every surface of Bonnet.

Bianchi has considered the class of surfaces defined by the

property: the spheres described on every segment of the normal comprised between the two centers of curvature as diameter cut a fixed sphere in great circles, or orthogonally, or all pass through the center. It is found that these surfaces are surfaces of Bonnet and their transforms also have the above property.

Bäcklund has shown that the transformation of Bianchi can be generalized so that for each angle σ there is an infinity of transforms of a pseudospherical surface such that the tangent planes to the given surface and the transform make the angle σ. This transformation also can be generalized so as to apply to all surfaces of Bonnet and the problem requires at most the solution of a Riccati equation. The points of contact of all transforms corresponding to a given σ lie in a circle whose axis is normal to S at M. If the problem has been completely solved for a given surface S, and if S_1 and S_2 are two of its transforms corresponding to angles σ_1 and σ_2, there is a transform of S_1 of angle σ_2 coinciding with a transform of S_2 of angle σ_1, and its determination requires only algebraic processes. From this we have that when all the transforms of a given surface can be found, the determination of the transforms of the transforms requires algebraic processes alone. These transformations are applied to particular surfaces of Bonnet and their transforms are considered.

3. Dini and Lie solved completely the problem of geodesic representation, which consists in the determination of all pairs of surfaces S, S_1 capable of point to point correspondence in such a manner that all the ∞^2 geodesics of S correspond to those of S_1. Dr. Kasner shows that the same surfaces are obtained if four systems of ∞^1 geodesics on each surface are to correspond. A more general problem arises, however, when merely three such systems are to correspond. (As to two systems, no problem arises, since then S and S_1 are entirely arbitrary.) This problem of *partial* geodesic representation is discussed for the conformal case and for the case where one of the surfaces is a plane, yielding extensions of recent results of Finsterwalder and Stäckel. The paper concludes with the explicit determination of all surfaces capable of conformal representation on the plane so that a system of (real) geodesics become straight lines.

5. The theorem given by Professor Osgood in the *Trans-*

actions, volume 2 (1901), page 273: "On a fundamental property of a minimum in the calculus of variations, etc." depends for its proof on the approximate determination of a limit for the value of the integral

$$\int_a^l [f'(x)]^2 dx, \qquad (a < l \leqq b);$$

more precisely, it is necessary to show that the value of the integral is greater than a certain positive quantity depending on a, b and $|f(l) - f(a)| = L$. Professor Goursat gives an exceedingly simple proof that

$$\int_a^l [f'(x)]^2 dx \geqq \frac{L^2}{l-a}.$$

6. Cosserat and Bianchi have given the necessary and sufficient condition that a surface should be susceptible of deformation so that a conjugate system of lines should remain conjugate during the deformation. Instead of a single surface, we may think of a continuous system of surfaces, which represent the different forms into which the single surface is deformed.

In Mr. Smith's paper, after proving Bianchi's theorem that surfaces which possess the property mentioned are in every case the associates of surfaces whose curvature, expressed in terms of the parameters of their asymptotic lines, is of the form

$$K = -\frac{1}{[\phi(u) + \chi(v)]^2}$$

two surfaces whose curvature is of this type are considered. They are the right conoid and a ruled surface of the third order, and it is shown that the deformation problem can be completely solved for these surfaces by considering their spherical representations. This leads to two classes of surfaces which may be deformed so that a conjugate system of lines remains a conjugate system; one of these, the associate of the conoid, was found by Goursat in 1891 by other methods, and the other has for its equations

$$x = \frac{\partial}{\partial u}\left[(U + V - vV') \cdot \frac{1 - 3u^2}{1 + u^2}\right] - 3V',$$

$$y = \frac{\partial}{\partial u}\left[(U + V - vV') \cdot \frac{1 - 3u^2}{1 + u^2}\right] + 3V',$$

$$z = \sqrt{2}\,\frac{\partial}{\partial u}\left[(U + V - vV') \cdot \frac{u(3 - u^2)}{1 + u^2}\right].$$

The properties of the latter surfaces are discussed and it is shown how the other members of the continuous system may be found by Bonnet's theory.

7. Professor Miller's paper is in the abstract as follows: According to Sylow's theorem there is only one set of conjugate subgroups of order p^α in a group G of order $p^\alpha n$, where p is any prime number and n is not divisible by p. The group G contains at least one subgroup of order p^β, $\beta < \alpha$. If m represents the number of sets of subgroups of order p^β which are contained in a particular subgroup of order p^α and are formed by combining into one set all of these subgroups of order p^β which are conjugate under G, then the number of the sets of conjugate subgroups of order p^β in G cannot exceed m. The number m can clearly not exceed the number of sets of conjugate subgroups of order p^β in one of the subgroups of order p^α. The latter number depends upon the type of the subgroup of order p^α, otherwise it is independent of the type of G.

In particular, if the subgroups of order p^α are cyclic, all the subgroups of order p^β which are found in G form a single conjugate set. Moreover, if a group of order p^α contains only one subgroup of order p^β, all the subgroups of order p^β in G form a single conjugate set. As β may equal α, this statement includes the fact that all the subgroups of order p^α form a single conjugate set under G. If $p^\alpha = 4$, G cannot contain more than three sets of conjugate subgroups of order 2. It is proved that there cannot be just two such sets of conjugate subgroups in G. If G contains contains three sets of conjugate operators of order 2 it must contain a characteristic subgroup of order m which is composed of all its operators of odd order.

8. Möbius is commonly stated to have proved the theorem that any point transformation of the plane which is a collineation is a projective transformation. An examination of Möbius's work, however, shows that, besides the assumption that we are dealing with a one to one point transformation which is

a collineation, certain other assumptions are tacitly made. The most obvious of these is the assumption that the transformation is continuous. Moreover, if we are to get anything beyond the affine transformations, we cannot assume that the transformation is defined at every point of both planes, since there are some points which will be "thrown to infinity" and therefore, strictly speaking, have no images in the other plane. It is the object of Mr. Swift's paper to establish this theorem of Möbius without the assumption of continuity and with as slight assumptions as possible concerning the regions for which the transformation is defined. The theorem proved takes the following form: Two plane point sets being given, subject to the sole restriction that each contains interior points, if it is possible to establish a one to one correspondence between the points of these sets of such a character that to any three collinear points of either set correspond three collinear points of the other, then this correspondence can be effected by a projective transformation. It should be noticed that here, as also in Möbius's work, no reference is made to imaginary points.

9. Miss Schottenfels's preliminary communication contains the proof that all types of simple groups of order $8!/2 = 20160$ separate, with respect to the number of conjugate subgroups of order 5, into two classes containing 336 or 2016 conjugate subgroups of order 5, respectively.

10. Professor Von Escherich in the *Denkschriften der Wiener Akademie*, volume 46, and Heffter in *Crelle's Journal*, volume 116, proved that there exists for linear differential homogeneous equations a theory analogous to that of algebraic equations, and this subject has recently been revived by Drs. Epsteen and Pierce in the *American Mathematical Monthly* of 1903.

The paper offered by Miss Schottenfels is supplementary to those of the last mentioned authors, and establishes the necessary condition that two linear homogeneous differential equations shall have two or more common integrals.

The theorem is illustrated by the following application to two linear homogeneous differential equations:

$$(1)\quad \alpha_0(x)\frac{d^4y}{dx^4} + \alpha_1(x)\frac{d^3y}{dx^3} + \alpha_2(x)\frac{d^2y}{dx_4} + \alpha_3(x)\frac{dy}{dx} + \alpha_4(x)\cdot y = 0,$$

$$(2)\quad \beta_0(x)\frac{d^3y}{dx^3} + \beta_1(x)\frac{d^2y}{dx_4} + \beta_2(x)\frac{dy}{dx} + \beta_3(x)\cdot y = 0.$$

The necessary condition that (1) and (2) have two integrals in common is that, in the following matrix obtained by successive differentiation,

$$\left\{\begin{vmatrix} \alpha_0 & \alpha_0'+\alpha_1 & \alpha_1'+\alpha_2 & \alpha_2'+\alpha_3 & \alpha_0'+\alpha_4 & \alpha_4' \\ 0 & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_0 & 2\beta_0'+\beta_1 & \beta_0''+2\beta_1'+\beta_2 & \beta_1''+2\beta_2'+\beta_3 & \beta_2''+2\beta_3' & \beta_3'' \\ 0 & \beta_0 & \beta_0'+\beta_1 & \beta_1'+\beta_2 & \beta_2'+\beta_3 & \beta_3' \\ 0 & 0 & \beta_0 & \beta_1 & \beta_2 & \beta_3 \end{vmatrix}\right\},$$

the determinant consisting of the first five columns, and also that consisting of the first four columns and the sixth, shall vanish identically.

F. N. COLE.

COLUMBIA UNIVERSITY.

TWO SYSTEMS OF SUBGROUPS OF THE QUATERNARY ABELIAN GROUP IN A GENERAL GALOIS FIELD.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, August 31, 1903.)

1. CONSIDER first the group G_ω composed of the

$$\omega = p^{4n}(p^{2n}-1)(p^n-1)$$

operators of the homogeneous quaternary abelian group in the $GF[p^n]$, $p>2$, which multiply the variable η_1 by a constant. Those of its operators which leave ξ_1 and η_1 unaltered are given the notation

$$\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}: \quad \begin{aligned} \xi_2' &= \alpha\xi_2+\gamma\eta_2, \\ \eta_2' &= \beta\xi_2+\delta\eta_2, \end{aligned} \qquad (\alpha\delta-\beta\gamma=1).$$

Certain other operators of G_ω are given the notation

$$[k, a, c, \gamma] = \begin{vmatrix} 1 & k & a & c \\ 0 & 1 & 0 & 0 \\ 0 & c-\gamma a & 1 & \gamma \\ 0 & -a & 0 & 1 \end{vmatrix}$$

They form a group of order p^{4n}, as shown elsewhere by the writer.* The general operator of G_ω may now be exhibited as either of the products

$$T_{1,\tau}[k, a, c, 0]\begin{bmatrix} a & \gamma \\ \beta & \delta \end{bmatrix}, \quad T_{1,\tau}\begin{bmatrix} a & \gamma \\ \beta & \delta \end{bmatrix}[k, -\beta c + \delta a, ac - \gamma a, 0].$$

Now $T_{1,\tau}T_{2,\sigma}$ transforms $[k, a, c, \gamma]$ into $[k\tau^2, a\tau\sigma^{-1}, c\tau\sigma, \gamma\sigma^2]$. The operators $[k, a, c, 0]$ are seen to form a group $G_{p^{3n}}$. The $T_{1,\tau}$ extend the latter to $G_{p^{3n}(p^n-1)}$. *Hence the groups* $G_{p^{3n}}$ *and* $G_{p^{3n}(p^n-1)}$ *are self-conjugate under* G_ω.

The quotient group $G_\omega/G_{p^{3n}(p^n-1)}$ may be taken concretely as the group $\Gamma_{p^n(p^{2n}-1)}$ of the binary substitutions $\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$. To a subgroup of order μ of the latter corresponds a subgroup of order $\mu p^{3n}(p^n-1)$ of G_ω. We obtain subgroups of order $d\mu p^{3n}$ of G_ω, where d is any divisor of p^n-1, by restricting τ in $T_{1,\tau}$ to marks which define a cyclic subgroup of order d of the cyclic group of all the p^n-1 operators $T_{1,\tau}$. If the subgroup of order μ is self-conjugate under Γ, the corresponding subgroup is self-conjugate under G_ω. This follows from the theory of isomorphism, or directly, since $[k, a, c, 0]$ transforms $\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$ into

$$[\gamma a^2 - \beta c^2 + \delta ac - aac,\ \beta c + aa - a,\ \gamma a + \delta c - c,\ 0]\begin{bmatrix} a & \gamma \\ \beta & \delta \end{bmatrix}.$$

2. As the first example, let $p^n = 3$. Then Γ is of order 24. Now Γ_{24} contains a self-conjugate Γ_2, a self-conjugate Γ_3, one set of 3 conjugate cyclic Γ_4, one set of 4 conjugate Γ_3, one set of 4 conjugate cyclic Γ_6, but no further subgroups.† Hence G_ω contains subgroups $G_{54\mu}$, $\mu = 1, 2, 3, 4, 6, 8$. Within G_ω, G_{54}, $G_{54\cdot 2}$ and $G_{54\cdot 8}$ are, therefore, self-conjugate, while $G_{54\cdot 3}$ and $G_{54\cdot 6}$ are self-conjugate only under $G_{54\cdot 6}$ and $G_{54\cdot 4}$ only under $G_{54\cdot 8}$.

3. Let next $p^n = 5$. Denote the general substitution of Γ_{120} by

$$S = \begin{bmatrix} a & \gamma \\ \beta & \delta \end{bmatrix}, \quad a\delta - \beta\gamma \equiv 1 \pmod 5. \tag{1}$$

If the characteristic determinant $D(k) \equiv k^2 - (a+\delta)k + 1$ of S is irreducible modulo 5, S is conjugate within Γ_{120} with one of the canonical forms ‡

* *Transactions*, vol. 4 (1903), pp. 371–386.

† Dickson, *Annals of Mathematics*, vol. 5 (1903–4).

‡ Dickson, *Transactions*, vol. 2 (1901), p. 117. Instead of $S_4' = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}$, we employ its transform S_4 by $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. In the $GF[5^2]$, S_3 and S_6 may both be given ultimate canonical forms of the type $\begin{bmatrix} k & 0 \\ 0 & k^{-1} \end{bmatrix}$, $k^6 = 1$.

$$(2) \qquad S_3 = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \quad S_6 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

The only substitutions of Γ_{120} commutative with S_6 are its powers

$$(3) \qquad \begin{bmatrix} \alpha & \gamma \\ -\gamma & \alpha+\gamma \end{bmatrix}, \qquad (\alpha^2 + \alpha\gamma + \gamma^2 \equiv 1).$$

The only ones commutative with S_3 are (3). If $D(k)$ is reducible modulo 5, S is conjugate within Γ_{120} with one of the canonical forms

$$(4) \qquad I, \quad S_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad S_4 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \qquad S_5 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$
$$S_{10} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad S'_{10} = \begin{bmatrix} -1 & 0 \\ -2 & -1 \end{bmatrix}, \quad S'_5 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

Indeed, S_{10} is conjugate with $\left[\begin{smallmatrix} -1 & 0 \\ -t & -1 \end{smallmatrix}\right]$ within Γ_{120} if and only if t is a quadratic residue of 5. Further,

$$(5) \quad S_{10}^2 = S'_5, \quad S_{10}^5 = S_2, \quad S_{10}^6 = S_5, \quad S_{10}^7 = S'_{10}, \quad S_{10}'^6 = S'_5.$$

Now S_4 is commutative only with its powers, and

$$\begin{bmatrix} \pm 1 & 0 \\ \pm t & \pm 1 \end{bmatrix} \text{ only with } \begin{bmatrix} \alpha & 0 \\ \beta & \alpha \end{bmatrix}, \qquad (\alpha^2 \equiv 1).$$

THEOREM: *The group Γ_{120} of all binary substitutions of determinant unity modulo 5 contains, in addition to the identity and the self-conjugate substitution S_2, one set of 20 conjugate substitutions of period 3, one set of 20 of period 6, one set of 30 of period 4, two sets each of 12 of period 5, and two sets each of 12 of period 10.*

The only substitutions of Γ_{120} which transform S_6 into its inverse are

$$(6) \qquad \begin{bmatrix} \alpha & \gamma \\ \alpha+\gamma & -\alpha \end{bmatrix}, \qquad (-\alpha^2 - \alpha\gamma - \gamma^2 \equiv 1),$$

each of period 4. Denote by Γ_{12} the group formed by the substitutions (3) and (6). Denote by Γ_{20} the group formed by the powers of S_{10} and the 10 substitutions $\left[\begin{smallmatrix} \pm 2 & 0 \\ \beta & \pm 2 \end{smallmatrix}\right]$ of period 4 which transform S_{10} into its inverse. Denote by Γ_8 the group of the powers of S_4 and the substitutions $\left[\begin{smallmatrix} 0 & \gamma \\ -\gamma^{-1} & 0 \end{smallmatrix}\right]$ of period 4 which

transform S_4 into its inverse. Since Γ_8 contains 3 groups of order 4 conjugate within Γ_{120}, it is self-conjugate under a subgroup Γ_{24}. Finally, denote by C_i the cyclic group generated by S_i.

We proceed to show that every subgroup of Γ_{120} is conjugate with one of the preceding. This has already been shown for the cyclic groups, and is evident for groups of orders 8, 20 and 40. A subgroup of order 10 is necessarily cyclic. A subgroup of order 12 contains 3 conjugate cyclic groups of order 4 and hence a single cyclic group of order 3, so that it is conjugate with Γ_{12}. A group of order 15 is cyclic and hence cannot be a subgroup. A subgroup of order 24 has 1 or 3 conjugate groups of order 8; if 1, it is conjugate with Γ_{24}; if 3, the group transforming each into itself is a self-conjugate subgroup of order 4 of the group of order 24, contrary to the above. There is no subgroup of order 30, since it would contain 6 cyclic G_5, 10 cyclic G_3, and 15 G_2. There is no subgroup of order 60, since it would contain 6 cyclic G_5, 10 cyclic G_3, and 15 cyclic G_4.

THEOREM.* *The subgroups of Γ_{120}, aside from itself and identity, are conjugate with C_2, C_3, C_4, C_5, C_6, Γ_8, C_{10}, Γ_{12}, Γ_{20}, Γ_{24}. Within Γ_{120}, the largest groups in which these are self conjugate are Γ_{120}, Γ_{12}, Γ_8, Γ_{20}, Γ_{12}, Γ_{24}, Γ_{20}, Γ_{12}, Γ_{20}, Γ_{24}, respectively.*

We may derive from §1 the following subgroups of G_ω:

$$G_{500\mu} \quad (\mu = 1, 2, 3, 4, 5, 6, 8, 10, 12, 20, 24).$$

Within G_ω, G_{500} and $G_{500\cdot 2}$ are self-conjugate, while the others are self-conjugate only under the respective groups

$$G_{500\lambda} \quad (\lambda = 12, 8, 20, 12, 24, 20, 12, 20, 24).$$

4. Consider next the group H_ω of homogeneous quaternary abelian substitutions† in the $GF[p^n]$, $p > 2$, subject to the condition that two variables η_1 and η_2 are replaced by linear functions of themselves

$$(7) \qquad \begin{bmatrix} \alpha_{11} & \gamma_{11} & \alpha_{12} & \gamma_{12} \\ 0 & \delta_{11} & 0 & \delta_{12} \\ \alpha_{21} & \gamma_{21} & \alpha_{22} & \gamma_{22} \\ 0 & \delta_{21} & 0 & \delta_{22} \end{bmatrix}$$

* By way of check, we note that the simple $G_{60} = \Gamma_{120}/C_2$ has subgroups only of the orders 2, 3, 4, 5, 6, 10 and 12, those of orders 6 and 10 being dihedral.

† Given by formula (19), without the sign $\pm$, page 380, *Transactions*, vol. 4 (1903).

If each $\gamma_{ij} = 0$, (7) becomes, in view of the abelian conditions,

$$(8) \qquad \begin{vmatrix} \delta_{22}/\Delta & 0 & -\delta_{21}/\Delta & 0 \\ 0 & \delta_{11} & 0 & \delta_{12} \\ -\delta_{12}/\Delta & 0 & \delta_{11}/\Delta & 0 \\ 0 & \delta_{21} & 0 & \delta_{22} \end{vmatrix}, \quad (\Delta = \delta_{11}\delta_{22} - \delta_{12}\delta_{21}).$$

The substitutions (8) form a group H_ρ of order

$$\rho = (p^{2n} - 1)(p^{2n} - p^n),$$

simply isomorphic with the group of all binary substitutions (of general determinant) in the $GF[p^n]$. Giving therefore to (8) the notation $\left(\begin{smallmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{smallmatrix}\right)$, we find that $\left(\begin{smallmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{smallmatrix}\right)[k, 0, c, \gamma]$ is identical with (7) if

$$\gamma_{11} = k\delta_{11} + c\delta_{21}, \qquad \gamma_{21} = c\delta_{11} + \gamma\delta_{21},$$
$$\gamma_{12} = k\delta_{12} + c\delta_{22}, \qquad \gamma_{22} = c\delta_{12} + \gamma\delta_{22}.$$

Inversely, every operator (7) may be obtained as such a product. Indeed, the four last conditions may be written

$$k\Delta = \begin{vmatrix} \gamma_{11} & \delta_{21} \\ \gamma_{12} & \delta_{22} \end{vmatrix}, \quad c\Delta = \begin{vmatrix} \delta_{11} & \gamma_{11} \\ \delta_{12} & \gamma_{12} \end{vmatrix}, \quad c\Delta = \begin{vmatrix} \gamma_{21} & \delta_{21} \\ \gamma_{22} & \delta_{22} \end{vmatrix}, \quad \gamma\Delta = \begin{vmatrix} \delta_{11} & \gamma_{21} \\ \delta_{12} & \gamma_{22} \end{vmatrix}.$$

These determine k, c and γ uniquely, the two values for c being equal in view of the abelian relation (C_{23} in the notation of Linear Groups, page 91)

$$\begin{vmatrix} \gamma_{11} & \gamma_{12} \\ \delta_{11} & \delta_{12} \end{vmatrix} + \begin{vmatrix} \gamma_{21} & \gamma_{22} \\ \delta_{21} & \delta_{22} \end{vmatrix} = 0.$$

By bringing the α_{ij} to the foreground in place of the δ_{ij}, we may reverse the order of the factors and give (7) the form

$$[k', 0, c', \gamma'] \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{pmatrix}, \quad \gamma_{11} = k'\alpha_{11} + c'\alpha_{12}, \text{ etc.}$$

Hence the commutative * group $K_{p^{2n}}$ of the substitutions $[k, 0, c, \gamma]$ is self-conjugate under H_ω. The quotient group may be taken concretely as H_ρ. To a subgroup H_τ of H_ρ corre-

* *Transactions*, vol. 4 (1903), formula (12), p. 377.

sponds a subgroup $H_{\gamma p^{2\alpha}}$ of H_ω. For the cases $p^\alpha = 3$ and $p^\alpha = 5$, we may apply at once the results of §§ 2 and 3.

5. The results thus far obtained may be applied to the simple group given as the quotient group of the homogeneous abelian group by the group composed of the identity and the substitution C which changes the sign of each variable. From G_ω and H_ω, we obtain $G'_{\frac{1}{2}\omega}$ and $H'_{\frac{1}{2}\omega}$, respectively. For $p^\alpha = 3$, $G'_{\frac{1}{2}\omega}$, namely G'_{648}, has subgroups $G'_{27\mu}$, $\mu = 1, 2, 3, 4, 6, 8$ (§ 2). For $p^\alpha = 5$, G'_{30000} has subgroups $G'_{250\mu}$, $\mu = 1, 2, 3, 4, 5, 6, 8, 10, 12, 20, 24$ (§ 3).

Let H_ρ correspond to $H'_{\frac{1}{2}\rho}$. For $p^\alpha = 3$, H'_{24} is simply isomorphic with the group of all linear fractional substitutions $\left(\frac{\delta_{11}\delta_{12}}{\delta_{21}\delta_{22}}\right)$ modulo 3. But the latter is simply isomorphic with the symmetric group G_{24} on four letters. The latter has the following non-conjugate subgroups, in addition to identity, itself, and the alternating group G_{12}:

$$G_2 = \{I, (13)\}; \quad G_3 = \{I, (123), (132)\};$$

$$G_6 = \{I, (123), (132), (12), (13), (23)\};$$

$$G'_2 = \{I, (13)(24)\}; \quad G_4 = \{I, (13), (24), (13)(24)\};$$

$$G'_4 = \{I, (12)(34), (13)(24), (14)(23)\};$$

$$G''_4 = \{I, (1234), (13)(24), (1432)\};$$

$$G_8 = \{I, (1234), (1432), (13)(24), (12)(34), (14)(23), (13)(24);$$

every subgroup being conjugate within G_{24} with one of the foregoing. Then * G_2 is self-conjugate only under G_4, G_3 only under G_6, G_6 only under G_6, G'_2 only under G_8, G_4 only under G_8, G'_4 only under G_{24}, G''_4 only under G_8, G_8 only under G_8, G_{12} only under G_{24}.

In passing to the subgroups of H'_{24}, it is convenient to make the letters 1, 2, 3, 4 correspond to the respective marks $1, -1, 0, \infty$. Then

$$(12) \sim z' = -z, \qquad (13) \sim z' = -z + 1, \qquad (34) \sim z' = 1/z,$$

$$(23) \sim z' = -z - 1, \quad (24) \sim z' = -z/(z+1), \quad (14) \sim z/(z-1).$$

* That the chosen groups, and not some of their conjugates, appear as the largest groups in which the types are self-conjugate is a result of the notation used.

THEOREM. *Every subgroup of H'_{24} is conjugate with H'_{24}, H_{12} (group of all of determinant $+1$), identity, or one of the following:*

$$H_2=\left\{I,\left(\begin{matrix}-1&1\\0&1\end{matrix}\right)\right\};\ H_3=\left\{\left(\begin{matrix}1&\delta\\0&1\end{matrix}\right)\right\};\ H_6=\left\{\left(\begin{matrix}\pm1&\delta\\0&1\end{matrix}\right)\right\};$$

$$H'_2=\left\{I,\left(\begin{matrix}-1&1\\1&1\end{matrix}\right)\right\};\ H'_4=\left\{I,\left(\begin{matrix}-1&1\\0&1\end{matrix}\right),\left(\begin{matrix}-1&1\\0&1\end{matrix}\right),\left(\begin{matrix}-1&1\\1&1\end{matrix}\right)\right\};$$

$$H_4=\left\{I,\left(\begin{matrix}0&-1\\1&0\end{matrix}\right),\left(\begin{matrix}\pm1&1\\1&\mp1\end{matrix}\right)\right\};$$

$$H''_4=\left\{I,\left(\begin{matrix}1&1\\1&0\end{matrix}\right),\left(\begin{matrix}0&1\\1&-1\end{matrix}\right),\left(\begin{matrix}-1&1\\1&1\end{matrix}\right)\right\};$$

$$H_8=\left\{I,\left(\begin{matrix}0&-1\\1&0\end{matrix}\right),\left(\begin{matrix}1&1\\1&0\end{matrix}\right),\left(\begin{matrix}-1&1\\0&1\end{matrix}\right),\left(\begin{matrix}-1&0\\1&1\end{matrix}\right),\left(\begin{matrix}0&1\\1&-1\end{matrix}\right),\left(\begin{matrix}\pm1&1\\1&\mp1\end{matrix}\right)\right\}.$$

Moreover, H_2 is self-conjugate only under H'_4; H_3 only under H_6; H'_2, H'_4 and H''_4 are self-conjugate only under H_8; H_6 and H_8 only under themselves; H_4 and H_{12} only under H_{24}.

To the preceding subgroups H_{12}, H_2, H_3, $\cdots$, H_8 of H'_{24} correspond subgroups $H_{27\cdot12}$, $H_{27\cdot2}$, $H_{27\cdot3}$, $H_{27\cdot6}$, $H'_{27\cdot2}$, $H'_{27\cdot4}$, $H_{27\cdot4}$, $H''_{27\cdot4}$, $H_{27\cdot8}$, respectively, of $H'_{\frac{1}{2}\infty}\equiv H'_{648}$. Within the latter, they are self-conjugate only under H'_{648}, $H'_{27\cdot4}$, $H_{27\cdot6}$, $H_{27\cdot6}$, $H_{27\cdot8}$, $H_{27\cdot8}$, H'_{648}, $H_{27\cdot8}$, $H_{27\cdot8}$, respectively.

THE UNIVERSITY OF CHICAGO,
July, 1903.

THE DETERMINATION OF THE CONSTANTS IN THE PROBLEM OF THE BRACHISTOCHRONE.*

BY PROFESSOR OSKAR BOLZA.

THE general solution of Euler's differential equation for the problem of the brachistochrone in a vertical plane is the doubly infinite system of cycloids†

$$\begin{aligned} x - x_0 + h &= \pm r(\omega - \sin\omega), \\ y - y_0 + k &= r(1 - \cos\omega), \end{aligned} \tag{1}$$

referred to a rectangular system of coördinates whose (x, y)-plane is the given vertical plane and whose positive y-axis is directed vertically downward; x_0, y_0 are the coördinates of the starting point A, k is a given constant, viz.,

$$k = v_0^2/2g,$$

where v_0 is the initial velocity and g the constant of gravity; finally h and r are the two constants of integration, and r is essentially positive.

We suppose that the endpoints A and B are fixed and propose *to determine the constants of integration so that the cycloid*

*Very little attention seems to have been paid to the question of the determination of the constants in the problem of the brachistochrone. I have been able to find only the following few references on the subject in the literature of the calculus of variations:

Johann Bernoulli who first proposed the problem of the brachistochrone in 1696, and Jacob Bernoulli give a geometrical construction for the special case where the initial velocity is zero, in which case the starting point is a cusp of the cycloid (compare Ostwald's Klassiker der exacten Wissenschaften, no. 46, pp. 12 and 18).

Dienger, Grundriss der Variationsrechnung (1867), p. 38, reduces for the general case the determination of the constants to the two transcendental equations (7) and (8) of the text without entering into a further discussion of these equations.

Weierstrass in his lectures (1882) states without proof that it is always possible to construct a cycloid upon a given base passing through two given points A and B, and only one cycloid which contains no cusp between A and B.

†Compare for instance Lindelöf-Moigno, Calcul des variations, p. 228, and Pascal, Die Variationsrechnung, § 31, where numerous historical references are given.

*shall pass through the two given points A and B, with the additional condition that no cusp * of the cycloid shall lie between A and B and that the angle ω increases as the point x, y describes the cycloid from A to B.*

We shall prove that this problem has always *one and but one solution* provided that

$$x_1 \neq x_0 \quad \text{and} \quad y_1 - y_0 + k \geqq 0. \tag{2}$$

If we denote by x_1, y_1 the coördinates of the endpoint B and by ω_0 and ω_1 the values of the angle ω at the two points A and B respectively, we have to determine the four quantities h, r, ω_0, ω_1, from the four equations

$$\begin{aligned} h &= \pm r(\omega_0 - \sin\omega_0), & x_1 - x_0 + h &= \pm r(\omega_1 - \sin\omega_1), \\ k &= r(1 - \cos\omega_0), & y_1 - y_0 + k &= r(1 - \cos\omega_1), \end{aligned} \tag{3}$$

with the additional inequality †

$$0 \leqq \omega_0 < \omega_1 \leqq 2\pi. \tag{4}$$

Since the function $\omega - \sin\omega$ always increases as ω increases, the upper sign must be taken when $x_1 > x_0$, the lower sign when $x_1 < x_0$.

Leaving aside the two special cases

$$v_0 = 0, \quad \text{and} \quad v_0 \neq 0, \quad y_1 = y_0,$$

in which the solution is rather trivial, we assume, in addition to the inequalities (2), that $v_0 \neq 0$ and therefore

$$k > 0, \tag{5}$$

and that

$$y_1 \neq y_0. \tag{6}$$

If we eliminate h and r between the four equations (3), we obtain

$$m[\omega_1 - \omega_0 - (\sin\omega_1 - \sin\omega_0)] = \cos\omega_0 - \cos\omega_1, \tag{7}$$

$$a \sin\frac{\omega_0}{2} = \sin\frac{\omega_1}{2}, \tag{8}$$

* An arc of a cycloid containing a cusp in its interior can never furnish a minimum for the time (Schwarz, Lectures, 1898–99, unpublished).

† If ω_0, ω_1, h, r is a solution of (4) and if μ is an integer, then $\omega_0 + 2\mu\pi$, $\omega_1 + 2\mu\pi$, $h \pm 2\mu\pi r, r$ is also a solution, furnishing however the same cycloid. Hence we may restrict ω_0 and ω_1, to the above interval without loss of generality.

where

$$m = \pm \frac{y_1 - y_0}{x_1 - x_0},$$

and

$$a = \sqrt{\frac{y_1 - y_0 + k}{k}},$$

the square root being taken positive.

According as $y_1 \gtrless y_0$, we have $m \gtrless 0$ and $a \gtrless 1$.

If we set

$$u = \frac{\omega_0 + \omega_1}{2}, \qquad v = \frac{\omega_1 - \omega_0}{2}, \tag{9}$$

we obtain from (8)

$$tg\,\frac{u}{2} = \frac{a+1}{a-1}\,tg\,\frac{v}{2}. \tag{10}$$

Hence if we introduce u and v in (7), express u in terms of v by means of (10) and put for brevity

$$z = tg\,\frac{v}{2}, \quad c = \frac{a-1}{a+1}, \quad b = \frac{c}{m}, \tag{11}$$

the problem reduces to the solution of the transcendental equation

$$F(z) \equiv \text{arc } tg\, z - \frac{z(c^2 + 2bz - z^2)}{(1+z^2)(c^2+z^2)} = 0. \tag{12}$$

On account of (4), the angle

$$\frac{v}{2} = \text{arc } tg\, z$$

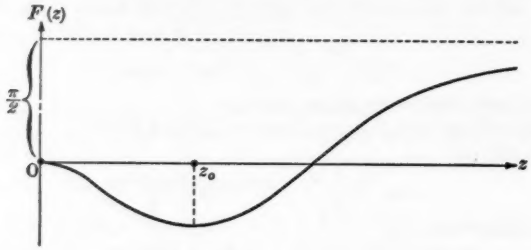
lies between 0 and $\pi/2$ and therefore z must be positive.

For the derivative of $F(z)$ we obtain

$$F'(z) = \frac{z[4bz^4 + 2(1+3c^2)z^3 + 2c^2(3+c^2)z - 4bc^2]}{(z^2+1)^2(z^2+c^2)^2}.$$

Since we suppose $x_0 \neq x_1$ and $y_0 \neq y_1$, m and c are both finite and different from zero and have the same sign, viz., the same sign as $y_1 - y_0$. Hence b is always positive. It follows therefore from Descartes's rule that the numerator of $F'(z)$ has *one and but one real positive root*, say $z = z_0$. Since moreover $F'(z)$

is negative for sufficiently small positive values of z, it follows that $F(z)$ decreases from the value zero to a certain negative minimum value, as z increases from 0 to z_0; then as z increases



from z_0 to $+\infty$, $F(z)$ increases continually and approaches for $z = +\infty$ the limit $+\pi/2$. Hence the equation (12) has one and but one real positive root.

This root being found, the equations (11), (10), and (9) yield a unique solution ω_0, ω_1, of (7) and (8), satisfying the inequality (4). Finally the values of h are z follow unambiguously from (3).

The existence and uniqueness of the solution of the proposed problem are therefore proved.

UNIVERSITY OF CHICAGO,
November 2, 1903.

ON THREE TYPES OF SURFACES OF THE THIRD ORDER REGARDED AS DOUBLE SURFACES OF TRANSLATION.

BY DR. A. S. GALE.

(Read before the American Mathematical Society, October 31, 1903.)

THIS note serves the double purpose of making a slight addition to the theory of three types of surfaces of the third order and of exhibiting the double surfaces of translation of lowest order. The latter surfaces enjoy all the properties of the double minimum surfaces * except those immediately dependent on the

* Lie, *Math. Annalen*, vol. 14 (1879), p. 346 et seq.; Darboux, Théorie des surfaces, vol. 1, p. 348 et seq.

minimum property; and as Henneberg's minimum surface of order fifteen is the simplest of the real double minimum surfaces, it is not uninteresting to see what very simple surfaces have many of the same properties.

By a double surface of translation is meant a surface whose equations have the form

$$S: \quad x = A(u) + A(v), \quad y = B(u) + B(v), \quad z = C(u) + C(v)$$

Such a surface may be generated in either one of two ways 1° by a curve

$$C: \qquad x = A(u), \quad y = B(u), \quad z = C(u),$$

subjected to the one-parameter system of translations

$$x' = x + A(v), \quad y' = y + B(v), \quad z' = z + C(v);$$

2° by the middle point of a chord of the curve

$$x = 2A(u), \quad y = 2B(u), \quad z = 2C(u).$$

If the surface S be subjected to a projective transformation, the transformed surface may evidently be regarded in two ways: 1° as the locus of a curve subjected to a one-parameter system of projective transformations of which the invariant configurations are surface elements with a common plane; 2° as the locus of the fourth harmonic to two points of intersection of a secant of a curve C and the point in which that secant meets a fixed plane.

Since two surfaces of the third order belong to the same type if one may be projected into the other, it is evident that these projective definitions will afford means of generating any surface of the third order if a surface of the same type may be regarded as a double surface of translation. The surface S will be of the third order, by Lie's theorem on the order of surfaces of translation,* if C is a twisted cubic; and if the axes be suitably chosen the elimination of u and v presents no difficulty, thus giving the equation of S in rectangular coördinates from which the nature of the singularities of S may be readily determined.

We find in this way that, if C cuts the plane at infinity in three distinct points P, Q and R, then S will be a surface of

* L. c., pp. 359, 354; see also Darboux, l. c., p. 369.

the third order with three conical points, C_2, situated at P, Q and R. If, however, C cuts the plane at infinity at P and is tangent to it at Q then the only singularities of S are a conical point C_2 at P and a binode B_5 at Q whose axis lies on S, one of the biplanes being tangent to S. Finally, if the plane at infinity is an osculating plane of C, then S is Cayley's ruled surface * of the third order.

The second projective definition then affords a simple means of generating any surface of these three types. When, however, C is a twisted cubic, that definition declares that S is the transformed of a plane by the involutory cubic transformation of space under which corresponding points lie on a secant of C and are harmonically situated with respect to the points of intersection of that secant with C. The above results have already been given from this point of view † but attention does not seem to have been called to the fact that the surface corresponding to the plane at infinity is a surface of translation.

Remarking that a projective transformation transforms a conjugate system of curves on a surface into a conjugate system on the transformed surface, the properties of double surfaces of translation afford the following theorem:

On the surfaces of the third order whose singularities are three conical points, or a conical point and a binode whose axis lies on the surface, one of the biplanes being tangent to the surface, and on Cayley's ruled surface of the third order, there is a singly infinite system of twisted cubics any one of which may be transformed into any other by a projective transformation of a system similar to a system of translations under a projective transformation T. Two of these cubics pass through each point of general position on the surface. These cubics form a conjugate system; they have an envelope into which any one may be transformed by a projective transformation similar under T to a homothetic transformation whose ratio is two. This envelope is the locus of points through which but one curve of the system passes and is an asymptotic line of the surface. The surface may be regarded as the locus

* Lie, l. c., footnote, p. 353.

† Cantonne, *Napoli Rendiconti*, vol. 25 (1886), pp. 186, 189 and 190. The transformation is studied by Reye, Geometrie der Lage, 3d ed., part 3, p. 129 and Schoute, *Nieuw Archief*, 2d series vol. 4 (1899). If we consider the transformation in which the cubic is replaced by any twisted curve, Lie's theorem on the order of a surface of translation affords the order of the surface corresponding to a plane.

of the fourth harmonic to two points of this envelope and the point of intersection of the secant through these two points with the plane of the singularities of the surface.

YALE UNIVERSITY,
October, 1903.

ON THE GENERATION OF FINITE FROM INFINITESIMAL TRANSFORMATIONS— A CORRECTION.

BY PROFESSOR H. B. NEWSON.

(Read before the Chicago Section of the American Mathematical Society, January 2, 1903.)

IN a paper entitled "Continuous groups of circular transformations" which the author read before this Society, April 24, 1897, and which was published in the BULLETIN (2) series, volume 4, pages 107–121, there occurs a serious error * which I desire to correct.

The error in question is a misstatement of the number of logarithmic spirals of the family $\rho = e^{(c+i)\theta}$ (where c is a parameter) that pass through a given point of the plane. It was stated on page 114 of the above mentioned paper that in general only two spirals of the family pass through a given point. In fact there are an infinite number of these spirals through a point P.

To show this let, the coördinates of P be $(\rho_1, \theta_1 + 2n\pi)$. Since $\rho_1 = e^{(c+i)(\theta_1+2n\pi)}$, then

$$\log \rho_1 \equiv \log r + 2im\pi = c(\theta_1 + 2n\pi) + i(\theta_1 + 2n\pi);$$

whence $\log r = c(\theta_1 + 2n\pi)$ or $c = \log r/(\theta_1 + 2n\pi)$. Since n is any integer, c may have any one of an infinite number of values. Thus there are an infinite number of spirals of the family through the point $(\rho_1, \theta_1 + 2n\pi)$. When $n = 0, 1, 2, 3, \cdots$ the corresponding spiral, starting from the origin, makes $0, 1, 2, 3, \cdots$ turns about the origin before passing through the point P.

The last paragraph on page 114 and the first on page 115, including theorems 7 and 8, of the above-mentioned article should be corrected to read as follows:

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* My attention was first called to this error by Professor Frank Morley.

Different values of c give us different spirals, each of which corresponds to a one parameter subgroup of G_{mn}. c varies continuously through all real values from $-\infty$ to $+\infty$, so that these spirals lie infinitely close to one another. They all pass through the unit point. As c approaches zero, the corresponding spiral approaches as a limit the circle of unit radius about the origin; as c approaches infinity the corresponding spiral approaches as a limit the straight line which is the axis of real numbers.

Every point in the plane not on the unit circle lies on an infinite number of discrete spirals, from which we infer that every loxodromic transformation in G_{mn} belongs to an infinite number of distinct one-parameter subgroups. Every hyperbolic transformation in G_{mn}, except the involutoric transformation, belongs to an infinite number of loxodromic subgroups as well as to the hyperbolic subgroup. The elliptic transformations in G_{mn} belong only to the elliptic subgroup. The involutoric transformation is common to the elliptic and the hyperbolic subgroups; the identical transformation is common to all the subgroups; the two pseudo-transformations are common to all except the elliptic subgroup. Any two loxodromic subgroups have an infinite number of discrete transformations in common.

THEOREM 7. *Every one-parameter subgroup in G_{mn} is continuous. Every non-elliptic transformation in G_{mn} belongs to an infinite number of distinct subgroups.*

This same geometric representation enables us to discuss intuitively the generation of finite transformations by the repetition of infinitesimal transformations. Every spiral passes through the unit point, and corresponding to the two points on the spiral adjacent to the unit point we have two infinitesimal transformations belonging to a one-parameter group. These are given by $k = e^{+(c+i)\delta\theta}$ and $k = e^{-(c+i)\delta\theta}$. The identical transformation divides the one parameter group into two portions, each of which contains an infinitesimal transformation. Every finite transformation in each portion of a one parameter loxodromic group can be generated by the repetition of the corresponding infinitesimal transformation. In the elliptic group for which the spiral reduces to a circle, every transformation can be generated from either elliptic infinitesimal transformation. In the hyperbolic group, for which the spiral reduces to the axis of real numbers, the transformations for which k is negative can not be generated by the repetition of either of the

hyperbolic infinitesimal transformations. Every non-elliptic transformation in G_{∞_3} can be generated from an infinite number of distinct infinitesimal transformations; for every such transformation belongs to an infinite number of distinct subgroups.

THEOREM 8. *Every finite transformation of the group G_{∞_3} can be generated by the repetition of an infinitesimal transformation of the group. Every elliptic (non-elliptic) transformation in G_{∞_3} can be generated from two (an infinite number of) distinct transformations. Hyperbolic transformations for which k is negative can not be generated by either hyperbolic infinitesimal transformation.*

UNIVERSITY OF KANSAS,
February, 1903.

STUDY'S GEOMETRY OF DYNAMES.*

Geometrie der Dynamen. Die Zusammensetzung von Kräften und verwandte Gegenstände der Geometrie. Von E. STUDY. Leipzig, Teubner, 1903. 8mo., 603 pp., 46 figs.

THE original purpose of Professor Study's book was to present a systematic geometric treatment of the composition of forces acting on a rigid body, but as the work progressed the idea broadened, resulting in an elaborate treatise on a whole field of geometry, hitherto nearly unknown.

Probably no other work on geometry that has appeared since the memoirs by Klein and by Lie in the early volumes of the *Mathematische Annalen* contains so many original and fruitful ideas as that under review.

The book is divided into three parts: the first (pages 1–122) treats of the composition of forces as a problem in pure geometry; the second (pages 123–225) treats the same problem analytically, making free use of symbolic notation and of line geometry; the third (pages 226–556) is devoted mainly to the discussion of a transformation first met with in the second part.

The first two parts contain a new geometric theory of the composition of forces and of infinitesimal motions. In the first part no use is made of other branches of mathematics than elementary geometry and trigonometry, but in order to express

* Dyname is the form used by Plücker in his English papers in the London Transactions Roy. Soc.

himself explicitly the author feels the necessity of introducing over a hundred new terms, which he defines. Apart from the use of so many new words, the matter is presented in a particularly attractive manner. The ideas of Wiener and of Clifford are systematized and generalized. The premises in each case are sharply defined and the theorems are so well stated that the exact meaning is easily grasped.

In order to express a motion as a line transformation, collineation and duality in a linear complex are first treated, then a new transformation is defined as follows: Let a line x be transformed by a general screw motion S into x'; this is indicated by

$$x(S)x'.$$

Associated with every pair of lines x, x' is another line x_1, namely, the bisector of the acute angle and of the common perpendicular of x, x'. If the lines x, x_1, x' be connected as follows:

$$x(\tau_1)x_1(\tau_2)x',$$

then

$$\tau_1\tau_2 = S = \tau_2\tau_1.$$

The operation τ is not a point transformation nor even a contact transformation, for lines which intersect go over into lines skew to each other. The operations of homography and duality can be expressed as linear transformations of line coördinates p_{ik}. By introducing a new number it was found that the operation τ could be expressed as a linear transformation of the p_{ik}. The problem then naturally presented itself to determine the meaning of the general linear transformation in the new coördinates.

The third part of the work, comprising about two thirds of the volume, is devoted to the discussion of this problem. In this part is derived a new metrical line geometry which in many respects differs widely from that of Plücker. Although no hint is given the reader regarding the origin of the new idea, it seems to the reviewer to be a sort of generalization of a paper by Klein,* and by the aid of a paper by E. Müller† to have a natural interpretation as follows: Given any conic

* "Ein Uebertragungsprinzip und der Pascalsche Satz," *Math. Annalen*, vol. 22.

† "Ein Uebertragungsprinzip des Herrn E. Study," *Archiv der Mathematik und Physik*, series 3, vol. 5, pp. 104–118.

$$x = \phi_2(\lambda), \qquad y = f_2(\lambda), \qquad t = \psi_2(\lambda)$$

in the plane $z = 0$. A straight line $ax + by + ct = 0$ will cut the conic in two points λ_1, λ_2. Let the (real or imaginary) values of λ be projected stereographically from the λ plane (or plane $z = 0$) on a sphere. The two points λ_1, λ_2 will always project into real points on the sphere, and therefore uniquely determine the real line joining them. In this way the ∞^4 real and imaginary lines of the plane $z = 0$ are put in (1, 1) correspondence with the ∞^4 real secants of a sphere. The fundamental elements of the latter are the ∞^2 tangents, since all the tangents of a pencil are images of a single tangent to the conic. Lines which are conjugate with respect to the conic go over into lines of space which intersect and which are conjugate with regard to the sphere. If the conic and the sphere each be taken as absolute in its respective space the theorem becomes:

Perpendicular lines in the plane go over into intersecting perpendicular lines of space.

The position of a line in the plane is defined by the (real or imaginary) coördinates a, b, c. If these numbers be transformed, 16 real parameters are involved, defining a group G_{16}. Perpendicularity (the property of being conjugate with regard to the fundamental conic) is preserved by this transformation. Thus far the image lines in space are only the real secants of a sphere; in order to include all the lines of space the radius of the sphere must be infinitely large. The fundamental points are now in the plane at infinity. In order to distinguish between the two points λ_i on the surface of the sphere and insure that the line connecting them does not lie entirely at infinity, it is convenient to think of all the lines of space being double, or arranged in two superposed sheets. The transformations will now also include similarity, making a G_{17}.

Among the more important invariants of this group, which can be established by the preceding considerations, the author mentions the following: (1) parallelism of two lines of the same sheet; (2) a finite line and its perpendicular infinite line; (3) skew orthogonality and (4) orthogonal intersection of finite lines of different sheets; (5) intersection of a finite line and an infinite line when belonging to different sheets; (6) orthogonal intersection of two infinite lines of different sheets; (7) three orthogonal transversals of a given line.

The development here traced is not mentioned by Professor Study, and perhaps he made no use of it; but it furnishes an easy way to understand the subsequent treatment.

In order to obtain the analytic expressions for the transformations of G_{17}, we shall reproduce a portion of the paper of Professor Müller. Let

$$K \equiv \Sigma a_{ik} p_{lm} = 0$$

be the equation of a linear complex. The direction vectors of its axis are $a_{12} : a_{13} : a_{14}$. If the invariant $A = \Sigma a_{ik} a_{lm}$ vanishes, the complex becomes special. In any case the coefficients a_{ik} may be called the coördinates of K.

If K_1, K_2 be two special complexes, such that

$$K = \lambda_1 K_1 + \lambda_2 K_2,$$

then the axes of K_1, K_2 are conjugate polars with regard to every complex K of the pencil. In particular, consider

$$K_1 = a_{12} p_{34} + a_{13} p_{42} + a_{14} p_{23} = 0,$$

$$K_2 = a_{34} p_{12} + a_{42} p_{13} + a_{23} p_{14} = 0.$$

The former is coaxial with K, the latter is a parallel complex. The complex

$$K' = a_{12} p_{12} + a_{13} p_{13} + a_{14} p_{14} = 0$$

is special; its axis is in the plane at infinity, perpendicular to the axis of K. If the special complex whose axis is the conjugate polar of this infinite line with regard to K be K'', then

$$K = \mu K'' + \nu K'.$$

Hence $K + \lambda K' = \mu K'' + (\nu + \lambda) K'$ always represents a line complex coaxial with K.

Consider the transformation with vanishing determinant

$$p_{14} = 0, \quad p_{42} = 0, \quad p_{34} = 0, \quad p_{12} = p'_{34}, \quad p_{13} = p'_{42}, \quad p_{23} = p'_{34}.$$

By means of this transformation any line through the origin, as

axis of a special complex, is transformed into a line at infinity perpendicular to it, and every line in the plane at infinity vanishes identically.

Any general complex K is transformed into a special complex whose axis is perpendicular to the axis of K and lies in the plane at infinity.

The transformation is designated by ϵ, so that

$$K\epsilon = \epsilon K = K', \quad \epsilon\epsilon K = \epsilon^2 K = 0,$$

hence

$$\epsilon^2 = 0.$$

With this further restriction, $\epsilon^2 = 0$, numbers of the form $m + m'\epsilon$ obey the ordinary laws of combination of complex numbers except that when $m = 0$, the number can not be used as a divisor. Numbers of this character are called *dual* numbers. When a complex K is multiplied by a dual number it goes over into a coaxial complex.

It is now possible to establish a correspondence between the lines of a sheaf whose vertex is the origin and the plane field of lines in the plane at infinity.

The coördinates x_1, x_2, x_3 of a line in a sheaf (or a plane) are defined as dual numbers of the form

$$x_1 = a_{12} + \epsilon a_{34}, \cdots$$

and the a_{ik} are regarded as coördinates of a complex. By multiplying a linear equation by a dual number of the form

$$a_{14}^2 + a_{42}^2 + a_{34}^2 - \tfrac{1}{2}A(a)\epsilon,$$

a special coaxial complex is obtained; hence it is no restriction to regard the a_{ik} as line coördinates.

In the finite part of space this geometry is now identical with Plücker's line geometry, but configurations at infinity are interpreted in an entirely different way. The various preëxistent ideas are skillfully brought into use in a strikingly elegant manner. One unconsciously gets the idea that the work of Hamilton, Clifford, Klein, Segre, Hankel, and others was conveniently designed for this particular purpose. Indeed a large part of the whole field of mathematics is shown to be a sort of stepping stone to line geometry. The admiration of the reader is dampened,

however, by an unfortunate self-conscious style that fails to do justice to the beauty of the subject. The mode of expression is frequently so involved that one not particularly sympathetic with the author's purpose will find the text very difficult reading. In this respect the third part of the book stands in striking contrast to the first part.

So much as to the method—a method that gives fair promise of very fruitful results. Now as to the applications which the author himself has made of it. The uses of the dual coördinates furnish the subject matter of about two thirds the entire volume. They are identical with Plücker's complex coördinates in the finite part of the plane, but are illusory for configurations at infinity. In order to represent a definite continuum, the author introduces new coördinates, which are called coördinates of the second kind, of the first natural system. They are defined by the equations

$$a_i = a_{ii} \qquad (i = 1, 2, 3)$$

$$a_{11} = \begin{vmatrix} a_{42} & a_{31} \\ a_{43} & a_{12} \end{vmatrix}, \quad a_{22} = \begin{vmatrix} a_{43} & a_{12} \\ a_{41} & a_{23} \end{vmatrix}, \quad a_{33} = \begin{vmatrix} a_{41} & a_{23} \\ a_{42} & a_{31} \end{vmatrix},$$

so that

$$a_1 a_{11} + a_2 a_{22} + a_3 a_{33} \equiv 0.$$

These coördinates are homogeneous in the sense that when a_i is replaced by ρa_i, a_{ii} by $\rho^2 a_{ii}$ the elements remain unchanged.

Finally a line at infinity is replaced by its pole with regard to the absolute circle in that plane. With these coördinates the line space forms a continuum in the same sense as projective geometry does, *i. e.*, by giving an added definition to infinite elements. Finally a third system of nine coördinates is defined by the equations

$$a_{ik} = a_i \cdot a_k,$$

which with the a_i make a system of twelve, between which several identities exist. These new coördinates are used to study configurations of limiting rays in the plane at infinity.

After these preliminary ideas the first application is a detailed and exhaustive classification of linear systems of complexes and the determination of their equations in the simplest form. Joined to this is a correspondingly full discussion of the surface

formed by the axes in a linear pencil of complexes, namely, the cylindroid and its various degenerations in particular cases. The congruence of axes in a linear sheaf of complexes receives too little attention compared with the length of treatment devoted to related subjects.

An equation of the second degree between the dual numbers x defines a congruence, since the real part (scalar) and the coefficient of ϵ (vector) must be separately equated to zero. Since the numbers x, when interpreted as coördinates in a sheaf, define a quadric cone, these congruences are called conical congruences.

The discussion of these congruences including the reduction of their equations to canonical forms is the subject matter of the interesting doctor dissertation of Dr. A. B. Pierce.* The paper of Dr. Pierce makes the various transformations of the G_{17} much easier to understand. Pages 1–19 of this paper are concerned with the ordinary dual coördinates, the canonical forms of the equations are given for all cases including the degenerate ones; pages 20–30 are devoted to a similar discussion of the coördinates of the second kind. The remainder of the paper treats of geometric interpretations.

An extensive discussion is given by Study of the congruence determined by

$$a_{41} = \sigma_1, \quad a_{42} = \sigma_2, \quad a_{43} = \sigma_3,$$

$$a_{23} = (p - e_1)\sigma_1, \quad a_{31} = (p - e_2)\sigma_2, \quad a_{12} = (p - e_3)\sigma_3,$$

and the linear sheaves of complexes obtained by varying p. The treatment is here projective; the author makes no use of his new coördinates. One can read this chapter (pages 461–534) without any knowledge of the preceding portion, although a fair knowledge of line geometry is presupposed. If the parameter p be unrestricted, the a_{ik} are Plücker complex coördinates, but if $A = 0$, the a_{ik} are tetrahedral line coördinates. First the case is considered in which

$$e_1 > e_2 > e_3, \quad \Sigma e_i = 0.$$

This configuration has been investigated by other writers.†

* "Classification and properties of dual conical congruences," Zürich, Meyer and Hendess, 1903. 8vo., 60 pp.

† See Ball's Theory of Screws, 2d edition, p. 170ff, W. Waelsch, "Transversalencongruenzen," *Wiener Berichte*, 1887, Part I. W. Stahl, "Das Strahlensystem dritter Ordnung und zweiter Classe," *Crelle's Journal*, vol. 91, where other references are given.

The congruence is of order 3 and class 2, dual to the congruence of axes in a linear sheaf of complexes. The discussion of the distribution of the lines on ∞^1 coaxial hyperboloids and their association to the conjugate congruences (*i. e.*, such that every complex of the first sheaf is in involution with all those of the second) is followed by a more thorough study of the parametric hyperboloids themselves. The pedal surface is coincident with the mean surface; it is a particular form of the Steiner surface. The focal surface is the same for both congruences; it is of order 6 and class 4. This surface is exhaustively discussed, first geometrically, then by means of elliptic functions.

After the treatment of the general case is completed, each particular case is fully discussed. In these 70 pages is found not only a systematic and far-reaching discussion of the configuration named, but also a large number of other problems suggested that might furnish themes for further investigation. The treatment differs essentially from that of other authors in the emphasis put upon particular and degenerate forms. While others confine their attention to the so-called "general case" — a case which in many applications does not exist at all — the present work concerns itself much more with each particular type, so that in the applications no exceptions are presented.

In conclusion we would say that the author has produced a work that will for some time furnish material for study and reflection. The style is not inviting, and the treatment is frequently involved, but the author has presented a new field of mathematics. Many commentaries will be necessary to make the work generally accessible, but they will surely come.

VIRGIL SNYDER.

CORNELL UNIVERSITY,
October 12, 1903.

ENCYKLOPÄDIE DER ELEMENTAR-MATHEMATIK.

Ein Handbuch für Lehrer und Studierende. Von HEINRICH WEBER und JOSEF WELLSTEIN. Erster Band. Elementare Algebra und Analysis. Von H. WEBER. Leipzig, Teubner, 1903. xiv + 447 pp.

IT was with much interest that scholars learned, a few years ago, of the proposed appearance of an Encyklopädie der Mathe-

matischen Wissenschaften, and this under circumstances to assure the highest standard of excellence. The publication of this monumental work was begun in 1898, and the parts which have thus far appeared have in general fulfilled the expectations of mathematicians. It has been international in its view and in its contributions, rich in its bibliography, and comprehensive in scope. It has, however, failed to touch the field of elementary mathematics in such a way as to be helpful to the great body of secondary teachers and students.

Recognizing the need for a more elementary work, the same publishers have undertaken the Encyklopädie der Elementar-Mathematik, originally planned for two volumes, but now announced to include a third on the applications of the subject.

The general scheme is quite as commendable as that of the more pretentious work in the higher fields, for students and teachers in elementary lines are even more in need of helpful suggestions as to bibliography, modern methods, and the improvement of their subjects from the standpoint of recent mathematics, than are the investigators in the advanced theory. A difficulty presents itself, however, in defining the term "Elementary mathematics," and here the authors confessedly lay themselves open to criticism. They admit that all attempts at fixing the limits on purely mathematical grounds have been unsatisfactory, and they have been compelled to resort to educational considerations instead. It must be confessed that the result has not been altogether satisfactory, and it is difficult to reconcile it with the announced position taken by the authors. Why analysis should include infinite series and products and the transcendental nature of e and π, but contain no reference to even the elements of differentiation and integration, is not apparent, and why so many of the common topics of the elementary mathematics found in every secondary school should be omitted altogether, is equally inexplicable.

The present volume is divided into three "books," the first treating of arithmetic, the second of algebra, and the third of analysis. The term arithmetic is used in the German sense, and thus includes what in other countries is quite generally designated as elementary algebra. Nearly two hundred pages are devoted to this subject, divided among the following topics: Natural numbers, the fundamental operations, fractions, irrational numbers, ratio, powers and logarithms, equations of the first and second degrees, imaginary numbers, permutations and combinations, and various applications to series.

The second book, on algebra, includes the following topics: Algebraic equations, the leading propositions of algebra, indeterminate linear and quadratic equations, continued fractions, the cubic and quartic equations, approximations of roots of numerical equations, cyclotomic equations, and a discussion of certain cases impossible of solution, as of the cubic by square roots alone, and of the quintic by radicals.

The book on analysis considers infinite series and products, including the question of convergency, and gives proofs of the transcendence of e and π.

The most interesting portions of the book on arithmetic are those devoted to the origin of the number concept, the theory of the negative, the Dedekind theory of the irrational, the Du Bois-Reymond theory of ratio, and the combinatory theory, including a few of the fundamental propositions of groups.

Several sections of the book on algebra are of especial interest. The existence theorem for roots, for example, is treated by the first (1799) of Gauss's methods (as improved in 1849), the presentation being exceptionally clear. Indeterminate linear equations form the basis for a brief treatment of number congruences, and indeterminate quadratics serve in a similar way for an introduction to the study of primes. The Galois theory of equations is presented in connection with the equation of the fourth degree, although no reference is made to its application to the quintic equation. The cyclotomic equation is especially well discussed, including Gauss's treatment of the regular 17-gon. The chapter on the impossibilities of algebra is also one of the most valuable in the whole work. One of the strange bits of conservatism is the author's failure to mention Horner's method of approximation, although, unlike many of the older continental writers, he is by no means ignorant of it.* Instead of making use of it in the Encyklopädie, he follows the usual German method, through Sturmian functions, the Regula falsi, and the Newtonian approximation.

As would naturally be expected, the work shows on every page the German accuracy of proof, and the name of Teubner is an assurance of a good piece of bookmaking. On these two points there can be no reasonable criticism. Indeed, the general appearance of the page is distinctly better than that in the

* Vide his Algebra, vol. I, p. 382.

larger Encyklopädie, even in respect to the mediocre figures. The latter are, however, below the general standard of the book, a capital in the figure sometimes appearing as a small letter in the text, a small x standing for both a point and a line, and the origin being indifferently O and o. Of misprints there seem to be few, although "solchen" (page 91) and "hypotonuse" (page 315) show that they exist.

From the outline already given it may rightly be inferred that the work is arranged and written rather in the style of a higher algebra than an encyclopædia in the English or French sense. It is certainly in no respect a *Lexikon*, even with its excellent index, and on the whole, by those who looked for a book of general information on elementary mathematics, it will be considered disappointing. But aside from the technical use of the name, which is quite correct from the Teutonic standpoint, one still has good cause for disappointment. He looks for a general and exhaustive treatise, and finds only a somewhat condensed higher text-book; he looks for a wide range of topics, and finds only a partial list of even the essentials; he looks for help in the way of bibliography, but he finds only a few commonplaces; he has hopes of an elementary work made up like Pascal's Repertorium, but he finds one more closely resembling Pund's Algebra. A teacher of the elements might reasonably expect, for example, to find some systematic treatment of factors in an encyclopædia of mathematics; he might hope for some suggestions on the subjects of graphs or undetermined coefficients; and he might feel that here he could have light thrown upon some of the vexing questions of extraneous roots, radical equations, and the exact meaning to be attached to certain symbols. He might also hope for some adequate treatment of elementary determinants, possibly a few suggestions as to the theory of numbers, and very likely a brief treatment of a topic like numerical computations. None of these topics will he find discussed, however, and some are not even mentioned.

It might also be expected that some efforts would be made to supply historical notes of interest to one who is teaching or reviewing the subject, even if not of value to one engaged in research. Such notes as appear are, however, practically worthless. Under complex numbers, for example, the names of the greatest contributors, save only Gauss, are wanting, while the only name mentioned in connection with the Delian

problem is Plato, and the assertion that Joost Bürgi first used a decimal point is altogether too emphatic. In connection with the cubic equation, Cardan's formula is referred to his Practica arithmeticæ generalis (1537) instead of the Ars magna (1545), an error due to misleading Cantor. Furthermore, not only are the historical references very meager, but they are confined almost exclusively to the German, and even then too exclusively to Cantor. For example, a work that pretends to be at all international might be expected to mention Heath's Diophantos in connection with a note upon the editions of this writer's works, though why these editions should be given at all in a work so barren of more important references, is a question.

A treatise of this kind might also be expected to furnish a good working bibliography, in no sense exhaustive, but suggestive and helpful. The bibliographical notes which Professor Weber has inserted are, however, with slight exception, of no practical value, and are evidently selected with no well defined purpose.

With all due appreciation of the scholarship of the work, and of its helpfulness, it must therefore be a matter of regret to all who have looked forward to its appearance, that the ground covered is not that of elementary mathematics in an international sense, that the historical notes are very ill considered, that no attempt has been made to offer a helpful bibliography, and that the arrangement and general treatment are so far removed from that of the Repertorium or the Burkhardt-Meyer Encyklopädie.

DAVID EUGENE SMITH.

SHORTER NOTICES.

Mathematical Papers of the late George Green. Edited by N. M. FERRERS and published by Macmillan in 1871. Facsimile reprint. Paris, A. Hermann, 1903.

GREEN'S celebrated paper on potential, published at Nottingham in 1828 by private subscription, remained practically unknown for many years. At this period George Green was entirely self-taught and had no more advantages than a provincial town with but few mathematical works of any kind was likely to furnish. He was, however, "discovered" and sent to Cambridge in 1833, taking his degree in 1837 and his fellow-

ship a couple of years later. He died in 1841, but before then had published some ten papers, each of which marked a definite advance. They were mainly applications of the celebrated theorem, known by his name, to various problems in wave motion. His college, feeling that his ability and publications had not met with the recognition they deserved, commissioned Ferrers to collect them in a volume which appeared in 1871. This so thoroughly fulfilled its object that for many years a copy has only been obtainable at second-hand book stores. The present edition is a facsimile reprint by a process reproduction, so that the collection will, owing to the enterprise of A. Hermann, be available to readers and libraries in exactly the original form. Most of the papers appeared in the *Cambridge Philosophical Transactions*, the early volumes of which are not found in a large number of American libraries. As might be expected, the printing has suffered slightly in the reproduction but the defect is in its general appearance; in no case will there be the least trouble from this cause to the reader.

ERNEST W. BROWN.

Problems in Astrophysics. By AGNES M. CLERKE. A. and C. Black, London, 1903.

THIS book has only an indirect bearing on mathematics and thus does not call for an extended notice in the BULLETIN. And yet it is one which the student of mathematical physics cannot altogether afford to neglect, for it contains much material which may be useful to him. Just as in bygone days the celestial motions furnished many problems to mathematicians and were indeed often the means of suggesting those problems, so in modern times the new astronomy has raised a new set of less exact and more difficult questions to be undertaken. As facts accumulate, and the phenomena begin to be grouped, the time comes when theories may be rightly broached, so that the problems enter within the range of the mathematician. He can find enough of them in this volume. The laws of rotation of the sun, the behavior of a mass of fluid or gas under its own attraction, the whole history of stellar development, the various forms of nebulæ, are some of them. The problem of the light curve of β Lyræ may perhaps belong to the older astronomy. The well-known work of G. H. Darwin on cosmical origins, and of Johnstone Ste on atmospheres are instances of the new methods, and Schuster

has shed a light on current ideas as to the existence of periodicity in various phenomena which may have the effect of clearing up many doubtful points in sunspot and other periods.

All these matters are here set forth. While giving in each case a concise story in which references to the original sources form a conspicuous element, Miss Clerke has at the same time made the volume eminently readable. It constitutes a valuable complement to her well-known work on the history of astronomy.

ERNEST W. BROWN.

Opinions et Curiosités touchant la Mathématique. (Deuxième Série.) Par GEORGES MAUPIN. Paris, C. Naud, 1902. 332 pp.

THE first volume of the Opinions et Curiosités appeared in 1898, and was reviewed in the BULLETIN, Volume 6 (1900), page 255. The present volume is made up of two parts. The first contains miscellaneous extracts from works mainly of the sixteenth and the seventeenth century, touching such subjects as the rotundity of the earth, the number of chemical elements, the nature of comets, the figure of the moon, ancient navigation, the capacity of Noah's ark, origin of the notion of the infinite, the squaring of the circle, etc.

The second part, covering 162 pages, is given up to biographical notes on Simon Stevin and Albert Girard, and to extracts from the Oeuvres Mathématiques of Stevin, edited by Girard and printed in 1634. The great historical importance of Stevin's book, as well as its rarity, justify this course. We find here quotations giving Stevin's argument "que l'unité est nombre," which are of value in tracing the development of the number concept. Several extracts exhibit Stevin's notation in decimal fractions and other parts of arithmetic. His great independence of thought is shown in his protest against calling incommensurable numbers like $\sqrt{8}$ "absurds, irrationels, irréguliers, inexplicables, sourds, etc." A number of physical and mechanical subjects are also touched upon, notably those relating to the tides, the formation of rain, the size of the earth, the center of gravity, the equilibrium of bodies.

The two parts of Maupin's book, taken together, make an interesting and entertaining volume. It is of value in exhibiting vividly certain stages in the progress of thought. It is good supplementary reading in a course on the history of mathematics and of physical science.

In former centuries mathematics was, in many minds, intimately interwoven with mysticism. A conspicuous example of this, given in Maupin's book, is the proof offered by Kepler, when he was a young man, that there cannot be more than six principal planets. He imagined the planets moving along great circles of concentric spheres so placed with respect to the five regular solids, taken in a specified order, that each solid was inscribed singly in one sphere and circumscribed about the next inner sphere. Since God created everything according to number and measure, these five regular solids determine the radii of six spheres in such a way that the radii are proportional to the distances of the planets from the sun. As there are no more than five regular solids and the number of spheres in the above arrangement cannot exceed six, there cannot be more than six principal planets.

In this early speculation Kepler is careless as to his assumptions, and does not allow himself to be controlled by the facts. Later, intercourse with Tycho Brahe and Galileo taught him the importance of accurate experimental data. Kepler's maturer reflection, in which his imagination was incessantly controlled and corrected by the facts in the case, ended in the brilliant discovery of "Kepler's Laws."

FLORIAN CAJORI.

Leitfaden der Projections-Lehre. Ein Uebungsbuch der konstruierenden Stereometrie. Von Professor Dr. CARL HEINRICH MÜLLER, Oberlehrer am Kaiser Friedrichs Gymnasium, Frankfurt a. M. und Professor OTTO PRESLER, Oberlehrer an der städt. Oberrealschule, Hannover. Ausgabe A. Leipzig, Teubner, 1903. 293 pp., 233 figs.

THE German student is taught the principles of drawing, perspective, and the elements of descriptive geometry before leaving the gymnasium. While not, on the whole, having as much of what is usually taught in America under the name of solid geometry, the average student who has completed the course in a realgymnasium or realschule has much more definite ideas of the visual properties of space than the average freshman in our colleges and technical schools.

Until recently, very few good books existed on constructive geometry, as it is taught almost entirely by use of a brief syllabus and a great deal of practice. The present volume is therefore somewhat of an innovation, representing the modern tendency to prepare text-books on all elementary mathematical subjects.

The book is divided into two parts, the first (pages 1–120) treating of oblique parallel perspective, and the second (pages 121–293) dealing principally with normal perspective. The first part begins with a number of simple problems of construction when the angle of incidence and amount of foreshortening are given. It is unfortunate that the authors of many of our books on solid geometry and on spherical trigonometry do not bear these simple rules in mind when preparing their figures. The principles are applied not only to the ordinary figures used in solid geometry but to much more complicated constructions of spherical astronomy, botany, zoölogy, physics and chemistry. The amount of material is not so extensive as that given in most works of descriptive geometry, but is amply sufficient for the mathematical student, and it is presented in such a form that one may readily extend it. A liberal allowance of unsolved exercises is added to each chapter.

The second part begins with a much more formal discussion of the elements of parallel perspective. The front, lateral, and ground elevations of a point are treated in connection with a large variety of positions defined by oblique perspective. The same detailed discussion is applied to a segment of a line, to a triangle, and to polygons and circles. The first space figures treated are the regular polyhedra and regular pyramids, then follow the cylinder and both cylindrical and conical spirals. A full discussion is given to an approximative method for drawing spherical loxodromes, orthographic maps, mercator's charts, etc., and a generous list of exercises furnishes practice for further constructions.

Unlimited lines and planes are next taken up, the treatment being similar to that given in most works on descriptive geometry. An interesting application is the construction of sun dials for any given latitude. Plane sections of surfaces and their development are then discussed as a natural application of the preceding principles. The problems regarding conic sections can not fail to awaken the student's interest.

The section devoted to curves of intersection of two surfaces is a particularly useful one. The part pertaining to spheroquartics is very attractive. A long section discusses shades and shadows in each of the fundamental planes, and gives the usual list of exercises for the student.

Finally, about fifty pages are devoted to central (conical) projection in connection with normal projection. The matter

is so arranged that it may be read immediately after the general introduction to each part, as no knowledge of the other portions of the book is presupposed. It would seem that this portion could have been materially improved by putting less emphasis on coördinates and more on the facts of collineation. It would have been still further improved by the addition of conical perspective of plane figures, and its relation to homology. In an appendix stereographic projection and conical mapping are defined and discussed. A second very interesting appendix shows the relation between the problems here treated and others of a more complicated nature. The historical growth of the subject is also outlined.

In the hands of a competent teacher the book will be of great service in assisting students to get clear ideas of the visual properties of space. VIRGIL SNYDER.

Catalog mathematischer Modelle für den höheren mathematischen Unterricht. Von MARTIN SCHILLING in Halle a. S. 6th edition, 1903. 8°, xvi + 130 pp.

THAT the so-called Steiner school of mathematics (to which the great geometer did not himself belong) is gradually being superseded by other systems is shown by the remarkable increase in both the number and variety of geometric models and apparatus.

In 1899 the well known Brill collection of models was transferred to Mr. Schilling and the whole enterprise was put upon a more systematic basis. Beside having the active support of the former contributors, the collection is now supervised and augmented under the scientific direction of Professor F. Schilling in Göttingen.

The present catalogue begins with a brief historical sketch, followed by a well-arranged table of contents from which one can readily determine exactly what is to be found in the collection of about 300 pieces. The catalogue proper consists of two parts; the first (pages 1–76) gives a short description of each model, and the series are arranged in the order in which they were produced; the second part contains a more technical description, often a figure, and the series are arranged according to subject matter. A detailed explanation is included in the memoir which accompanies each model sent to purchasers.

The most important additions which have been made during the last five years are the following:

Series XXIV. Twelve kinematic models by Professor F. Schilling illustrating the generation of epitrochoids and hypotrochoids with fixed and with clamped centers, ellipses, cycloids and the inversors of Peaucellier, of Hart, and of Kempe. They are made of brass and celluloid and mounted on wood which has the generated curve traced on its surface (marks 545).

Series XXV. Seven thread models by Professor H. Wiener of the forms of cubic cones according to the classification of Möbius (marks 128). The accompanying memoir (not contained in the catalogue) of 34 pages contains an excellent synthetic discussion of the forms of plane cubic curves.

Series XXVI. *A*, ten laboratory pieces by Professor G. Hauck and *B*, eighteen special pieces by Professor F. Schilling, for illustration in descriptive and projective geometry (marks 264). The second group includes an ingenious apparatus for generating an ellipse by means of two projective pencils of lines.

Series XXVII. Three stereometric wire models of electric lines of potential and lines of force, by Professor O. Wiener (marks 200).

Series XXVIII. Six models of the twisted cubic curve, by Dr. W. Ludwig (marks 160). In this series is included an excellent thread model of the quartic developable surface.

Series XXIX. Three models to illustrate the theory of the top, by Professor H. Grassmann (marks 265).

In the buildings of the university at Göttingen ample and well equipped rooms are set apart for the construction and exhibition of models. Model-making is not only a fine art but an important discipline.

VIRGIL SNYDER.

NOTES.

THE fourth regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of California on Saturday, December 19. A report of the meeting will appear in an early number of the BULLETIN.

THE staff of associate editors of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY, as constituted for the year 1904, will consist of Professor E. B. VAN VLECK, Professor H. S. WHITE, Dr. C. L. BOUTON, Professor L. E. DICKSON,

Professor J. I. HUTCHINSON, Dr. EDWARD KASNER, and Dr. E. B. WILSON.

THE issue of the January number of the *American Journal of Mathematics* will be somewhat delayed, owing to a serious fire in the printing office.

AN ANNUAL conference of academies and high schools was held at the University of Chicago on November 13–14. Professor H. E. SLAUGHT was chairman of the mathematical section. A paper on "The teaching of elementary mathematics from the standpoint of the teacher of physics" was presented by Dr. E. C. WOODRUFF.

THE Central association of science and mathematics teachers held its third meeting at the Northwestern University Professional School in Chicago, November 27–28. The following papers were presented before the mathematical section: "A partial presentation of some experimental work correlating mathematics and physics, by Professor G. W. MYERS; "What mathematics shall be taught in the eighth school year?", by Professor G. H. HOWE; "Current events of interest to teachers of mathematics," by Professor C. E. COMSTOCK; "What mathematics has the teacher of physics a right to expect of his students?", by Professor W. J. RISLEY. Professor MYERS exhibited some inexpensive and easily handled apparatus for illustrating problems in algebra, especially in solving equations and in variation. Professor C. E. COMSTOCK was elected chairman of the section.

THE Association of teachers of mathematics in the Middle States and Maryland was formally organized at Columbia University on November 28, 1903, with a membership of about 250. The programme announced in the December BULLETIN was carried out, and the following officers were elected for the ensuing year: President, Professor D. E. SMITH; Vice-president, Professor H. B. FINE; Secretary and Treasurer, Dr. ARTHUR SCHULTZE.

THE recommendations made by the committee on college entrance requirements of the AMERICAN MATHEMATICAL SOCIETY have been adopted by the College entrance examination board and will be made the basis of the board's examinations in mathematics on and after June, 1905.

THE National academy of sciences held its annual meeting at Chicago on November 17–19. No mathematical papers were presented.

AT the annual meeting of the London mathematical society held on November 12, the following papers were read: By Mr. W. H. YOUNG, "On sequences of sets of intervals containing a given set of points"; by Mr. H. HILTON, "On spherical curves"; by Dr. H. F. BAKER, "On the Weddle quartic surface"; by the Rev. F. H. JACKSON, "A formal generalization of Maclaurin's theorem"; by Mr. W. H. JACKSON, "Diffraction"; by Mr. H. S. HARDY, "A general theorem concerning absolutely convergent series"; by Professor J. D. EVERETT, "Note on Borgnet's method of dividing an angle in an arbitrary ratio"; by Mr. E. T. WHITTAKER, "On an expression of the electromagnetic field by means of two scalar potential functions"; by Professor A. E. H. LOVE, "The propagation of wave-motion in an isotropic elastic solid medium"; by Professor R. W. GENESE, "Notes on quaternions, including a simple construction for $Va\beta\gamma$."

UNIVERSITY OF PARIS. — The following mathematical courses are given during the first semester of the present academic year, opening November 9, 1903: — By Professor G. DARBOUX: Higher geometry, two hours. — By Professor E. GOURSAT: Advanced calculus and elements of analytic functions, two hours. — By Professor P. PAINLEVÉ: Mechanics of equilibrium and of motion, two hours. — By Professor P. APPELL: General introduction to higher mathematics, with particular regard to the applications, two hours. — By Professor H. POINCARÉ: Perturbations of the planetary orbits, two hours. — By Professor G. KOENIGS: Kinematics, two hours. — By Professor J. BOUSSINESQ: Thermomechanical properties of fluids and solids, two hours. Mathematical conferences will be held as follows: — By Professor S. RAFFY: Infinitesimal geometry and calculus. — By Professor P. PUISEUX: Mechanics. — By Professor H. ANDOYER and Mr. BLUTEL: General conferences for students of mathematics.

TEACHERS in the lycées of France must have passed the state examination (l'agrégation), which is independent of any particular school or college. The Normal School in Paris was founded to prepare students for these competitive examinations, and the quality of the scholarship of the candidates from this

institution has been such that a very high standard in the examinations has been maintained. Of late however, both the Normal School and the University of Paris have been seriously trenching on each other's field of work. In order to adjust this condition and to effect a unity of effort between the two institutions, the Normal School is to be intimately affiliated with the University after November 1, 1904.

THE Royal Institute of Venice announces the following prize problem:

"To complete in some important point the projective geometry of algebraic surfaces of two dimensions in space of n dimensions."

Competing memoirs may be written in Italian, French, German, or English and must be sent to the secretary of the Institute, under the usual conditions, before December 31, 1906. The amount of the prize is 3000 lire.

THE Accademia Pontaniana of Naples announces as its prize subject for 1904: "An important contribution to the general intrinsic theory of curves."

THE metric system has been adopted by New Zealand, to be exclusively employed from January, 1906.

TWO new lectureships in mathematics, of the value of £200 a year, have been established at the University of Cambridge by an unknown donor. They are to be known as the Cayley and the Stokes lectureships respectively, and have been filled by the appointment of Dr. E. W. HOBSON, of Christ's College and Dr. H. F. BAKER, of St. Johns.

PROFESSOR HUDSON'S lectures on the teaching of mathematics at King's College, Cambridge, have been postponed to next term, beginning January 23, 1904.

PROFESSOR J. LARMOR has been nominated by the council as one of the Secretaries of the London royal society.

PROFESSOR G. DARBOUX, L. BOLTZMANN, and H. POINCARÉ have accepted the official invitation to deliver addresses before the mathematical department of the congress of arts and science to be held in connection with the St. Louis exposition.

AT the University of Paris, Professor P. PAINLEVÉ has been appointed to the newly established chair of mathematics, and Professor H. ANDOYER has been transferred to the chair of physical astronomy.

PROFESSOR H. POINCARÉ has been elected a corresponding member of the Vienna academy of sciences.

THE honorary degree of doctor of science was bestowed upon Lord KELVIN on November 13, 1903, by the University of Wales.

DR. G. W. HILL has been elected a corresponding member of the Paris academy of sciences, section of astronomy, to succeed Professor G. SCHIAPARELLI, who has been elected a foreign associate of the Academy.

PROFESSOR M. SIMON, of the Lyceum at Strassburg, has been appointed honorary professor of mathematics in the University of Strassburg.

PROFESSOR J. CLAIRIN, of Dijon, has been appointed master of conferences in the University of Lille.

PROFESSOR R. LE VAVASSEUR, of Toulouse, has been appointed master of conferences in the University of Lyons, to succeed Professor E. CARTAN, who has been called to a similar position in the University of Nancy.

PROFESSOR J. DRACH, of Lille, will deliver a course of lectures on rational mechanics at the University of Poitiers during the present semester.

PROFESSOR E. LACOUR, of Nancy, has been appointed professor of pure mathematics in the University at Rennes.

MR. B. HOPKINSON, of Trinity College, has been elected professor of mechanism and applied mechanics at Cambridge University, in place of Professor J. A. EWING, resigned.

DR. K. PETR, of the Bohemian technical high school at Brünn, has been appointed associate professor of mathematics in the Bohemian University of Prague.

DR. G. B. HALSTED has been appointed professor of mathemathics at Kenyon College, Gambier, Ohio.

DR. H. C. DE MOTT has been appointed professor of mathematics in the Illinois Wesleyan University.

THE following academic appointments are also announced: Mr. W. R. RANSOM and Mr. E. E. FROELICH as instructors in mathematics in Harvard University; Mr. W. E. MACDONALD and Mr. B. H. CAMP as instructors in mathematics

in the Massachusetts Institute of Technology; Mr. F. E. KNOWLES as instructor in mathematics in the University of Oklahoma; Mr. W. O. BEAL as instructor in mathematics in Illinois College; Mr. O. S. STETSON as instructor in mathematics in Syracuse University; Mr. J. W. BRISTER as associate professor of mathematics in the University of Nashville; Mr. M. A. MACKENZIE, of Trinity College, Toronto, as associate professor of mathematics in the University of Toronto.

THE following catalogues of second-hand mathematical works have recently been issued: B. Quaritch, 15 Piccadilly, London, catalogue No. 225, 114 numbers, mostly very rare works; Theodor Ackermann, Promenadenplatz 10, Munich, catalogue 175a, 2 pages; J. Halle, Ottostrasse 3a, Munich, catalogue 33, 317 works on mathematics, mostly rare early editions, also 443 works on applied mathematics; Anton Creutzer, Aachen, catalogue 82, mathematics and astronomy; F. L. Dames, Landgrafenstrasse 12, Berlin W., catalogue 81, mathematics and physics; Max Weg, Leplaystrasse 1, Leipsic, catalogue 91, 1581 numbers, including works in applied mathematics and 129 portraits.

MARCUS BAKER, cartographer of the U. S. Geological Survey and assistant secretary of the Carnegie Institution, died at Washington, December 12. Mr. Baker had been a member of the AMERICAN MATHEMATICAL SOCIETY since 1891.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BALL (R. S.). On the reflection of screw-systems and allied questions. Dublin, 1903. 4to. (*Transactions of the Royal Irish Academy*, Vol. 32, section A, part 6, pp. 101–154.) 1s. 6d.

BLICHFELDT (H. F.). On the order of linear homogeneous groups. (*Transactions of the American Mathematical Society*, Vol. 4, pp. 387–397.)

BOOLE (M. E.). Lectures on the logic of arithmetic. Cambridge, Macmillan and Bowes, 1903. 8vo. Cloth. 2s.

BOWDEN (J.). Elements of the theory of integers. New York, Macmillan, 1903. 12mo. 10 + 258 pp. Cloth. $1.25

BUGAIEV (N. V.). Introduction to analysis and to the differential calculus. Moscow, 1902. 8vo. 204 pp., 1 plate. (Russian.)

FIEDLER (W.). See SALMON (G.).

FINK (E.). Eliah Wilna und sein elementar-geometrisches Compendium. Frankfurt a. M., 1903. 8vo. 29 pp. M. 1.50

GRACE (J. H.) and YOUNG (A.). Algebra of invariants. London, Clay, 1903. 8vo. 392 pp. Cloth. 10s.

GRODSKY (G. D.). Text-book of analytic geometry. Part I: Plane analytic geometry. St. Petersburg, 1903. 8vo. 210 pp. (Russian.) M. 4.00

HAGEN (J. G.). Synopsis der höheren Mathematik. Vol. III: Differential- und Integralrechnung. 4te Lieferung. Berlin, Dames, 1903. 4to. Pp. 129–256. M. 5.00

HARTWIG (T.). Transformation sphärischer Punkt- und Linienkoordinaten. (Progr.) Wiener-Neustadt, 1903. 8vo. 22 pp.

HELLER (T.). Der einfachste Körper, welcher kinetische Geometrie besitzt. (Progr.) Nürnberg, 1903. 8vo. 41 pp.

HOCHHEIM (A.). Aufgaben aus der analytischen Geometrie der Ebene. Heft I: Die gerade Linie, der Punkt, der Kreis. A. Aufgaben. 3te, vermehrte Auflage, bearbeitet von F. Hochheim. Leipzig, Teubner, 1904. 8vo. 6 + 98 pp. Cloth. M. 2.40

HOULLEVIGNE (L.). Rapport sur le progrès scientifique (mathématiques, physique, chimie), présenté aux assises littéraires et artistiques (3e session, Caen, 4–6 juin 1903). Paris, Valin, 1903. 8vo. 43 pp.

HUSS (J.). Ueber eine Anwendung der Theorie der linearen Differentialgleichung der Kugelfunktionen einer Variablen. (Diss.) Greifswald, 1903. 8vo. 16 pp.

JESSOP (C. M.). A treatise on the line complex. London, Clay, 1903. 8vo. 380 pp. Cloth. 10s.

JONES (A. C.). Notes on analytical geometry: an appendix. London, Frowde, 1903. 8vo. 176 pp. Cloth. 6s.

KRUG (A.). Die lineare Differentialgleichung dritter Ordnung. (Progr.) Aussig, 1903. 8vo. 60 pp.

LINNEBORN (A.). Die Fokaleigenschaften der Gebilde zweiter Ordnung in der Riemannschen Raumform. Münster, 1902. 8vo. 26 pp. M. 1.80

MERCHICH (M.). De veris geometriae integrae principiis contra geometras Euclideos simul et Noneuclideos. Zagrabiae, Merchich, 1903. 8vo. 37 pp. M. 1.00

MILLER (G. A.) and MORENO (H. C.). Non-abelian groups in which every subgroup is abelian. (*Transactions of the American Mathematical Society*, Vol. 4, pp. 398–404.) New York, 1903. 8vo.

MORENO (H. C.). See MILLER (G. A.).

RÜBENSTEIN (N.). Ueber Darstellung von Funktionen durch periodische Reihen. (Progr.) Mähr.-Ostrau, 1903. 8vo. 40 pp.

SALMON (G.). Analytische Geometrie der Kegelschnitte mit besonderer Berücksichtigung der neueren Methoden. Frei bearbeitet von W. Fiedler. 6te Auflage. Teil 2. Leipzig, Teubner, 1903. 8vo. 24 pp. and pp. 443–854. M. 8.00

SEYBOLD (C.). Die Drusenschrift Kitab Alnoqat Waldawair (Das Buch der Punkte und Kreise). Nach dem Münchener und Tübinger Codex. Tübingen, 1902. 4to. 111 pp. M. 4.00

SOCHOCKI (I.). Zasady teoryi funkcyj eliptycznych. [Elements of the theory of elliptic functions.] Warszawa, 1903. 8vo. 50 pp. R. 0.50

STOFFAES. Cours de mathématiques supérieures, à l'usage des candidats à la licence ès sciences physiques. Paris, Gauthier-Villars, 1904. 8vo. 7 + 538 pp. Fr. 10.00

WELD (L. G.). A short course in the theory of determinants. 3d edition. New York, Macmillan, 1903. 12mo. 13 + 238 pp. Cloth. $1.90

WOLLETZ (K.). Ueber eine von Jakob Steiner aufgestellte birationale geometrische Verwandschaft vom zweiten Grade nebst Anwendung. (Progr.) Jägernlorf, 1903. 8vo. 39 pp., 1 plate.

YOUNG (A.). See GRACE (J. H.).

II. ELEMENTARY MATHEMATICS.

AMALDI (U.). See ENRIQUES (F.).

APEL (O.). Ueber die Behandlung einiger mathematischer Kapitel im Unterricht. (Progr.) Halle, 1903. 4to. 20 pp. M. 1.20

AUGUST (E. F.). Logarithmische und trigonometrische Tafeln. 25ste Auflage, von F. August. Leipzig, 1903. 12mo. 8 + 204 pp. Boards. M. 1.60

BAUDOIN (P.). Leçons d'arithmétique et d'algèbre, à l'usage des classes de quatrième B et de troisième A des lycées et collèges, et de tous les établissements d'instruction. Paris, André, 1903. 8vo. 232 pp.

BOREL (E.). Algèbre (second cycle). Paris, Colin, 1903. 16mo. 7 + 379 pp. Fr. 3.00

C. (F. I.). Eléments d' algèbre, avec de nombreux exercises. Paris, Poussielgue, 1903. 16mo. 8 + 413 pp.

CERAMICOLA (F.). Saggio di geometria intuitiva ad uso dei ginnasi inferiori. Recanati, 1903. 8vo. 77 pp., 8 plates.

CORTÁZAR (D. J.). Tratado de trigonometría y topografía. 22a edición, corregida y arreglada por D. J. de Cortázar. Madrid, Perlado, 1904. 8vo. 216 pp. Fr. 4.50

DUPORT (J. B.). Lehrbuch der Arithmetik für die II. und III. Klasse der Mädchen-Lyzeen. Wien, Deuticke, 1903. 8vo. 3 + 175 pp. Cloth. M. 2.25

——. Lehrbuch der Arithmetik und Geometrie für die 6te Klasse der Mädchen-Lyzeen. Wien, Deuticke, 1903. 8vo. 4 + 217 pp. Cloth. M. 3.00

DUPUIS (J.). Tables de logarithmes à sept décimales. Edition stéréotype, contenant les logarithmes des nombres de 1 à 100,000, les logarithmes des sinus et des tangentes des angles calculés de seconde en seconde pour les cinq premiers degrés et de dix secondes en dix secondes pour tous les degrés du quart de cercle, et quelques

tables usuelles. 12e tirage. Paris, Hachette, 1903. 8vo. 12 + 580 pp. Fr. 3.50

ENRIQUES (F.) e AMALDI (U). Elementi di geometria elementare ad uso dei ginnasi superiori. Bologna, Zanichelli, 1903. 16mo. 136 pp.

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GRAF (H. G.). Wie lehrt und lernt man die Raumlehre? Ein Handbuch für Lehrer und Schüler. Heft 1. Neuwied, Heuser, 1904. 8vo. 8 + 60 pp. M. 1.20

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HECHT (B.). Zusammenstellung des mathematischen Lehrstoffes für die untere und mittlere Stufe des Realgymnasiums. Königsberg, 1903. 8vo. 52 pp. M. 1.60

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IRIBARREN Y ARCE (L.). See GÓMEZ PALLETE (D. J.).

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MARTUS (H. C. E.). Mathematische Aufgaben zum Gebrauche in den oberen Klassen höherer Lehranstalten. (In 4 Teilen.) Teil II: Ergebnisse der Aufgaben in Teil I. 11te Auflage. Dresden, 1903. 8vo. 280 pp. M. 4.80

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SAMPERE Y CARRERAS (J.). Elementos de algebra, conteniendo el cálculo de probabilidades y tablas logarítmicas. Valladolid, Santarén Madrazo, 1903. 8vo. 296 pp. Fr. 7.00

SHMULEVICH (P. K.). Collection of examination problems in higher schools. Part I: Trigonometry. 4th edition. St. Petersburg, 1903. 8vo. 214 pp., 1 plate. (Russian.) M. 4.50

SUÁREZ SOMONTE (I.). Nociones de geometría. Madrid, Méndez, 1903. 8vo. 244 pp. Fr. 11.00

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III. APPLIED MATHEMATICS.

BASHFORTH (F.). Historical sketch of the experimental determination of the resistance of the air to the motion of projectiles. Cambridge, 1903. 8vo. 30 pp., 1 plate. 1s.

BIENECK (J.). Ueber die astigmatische Abbildung einer unbegrenzten Ebene in einer Kugel bezüglich eines festen Augenpunktes. (Diss.) Rostock, 1903. 8vo. 31 pp., 2 plates.

BRUNNER (E.). Reaktionsgeschwindigkeit in heterogenen Systemen. (Diss.) Göttingen, Vandenhoeck & Ruprecht, 1903. 8vo. 66 pp. M. 1.80

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COYM (A.). Ueber elastische Schwingungen, die durch Austrahlung von Energie gedämpft werden. (Diss.) Berlin, 1903. 8vo. 58 pp.

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EICHLER (M.). Beiträge zur Theorie der astigmatischen Abbildungen von Objekten in hyperbolischen Spiegeln. Mit Anwendung auf die Abbildung des Sternhimmels. (Diss.) Rostock, 1903. 8vo. 52 pp. 6 plates.

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HAWKINS (C. C.) and WALLIS (F.). Dynamo: its theory, design, and manufacture. 3d edition, revised and enlarged. London and New York, Macmillan, 1903. 12mo. 13 + 925 pp. Cloth. $3.00

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MURANI (O.). Onde hertziane e telegrafo senza fili. Milano, Hoepli, 1903. 16mo. 15 + 341 pp. (Manuali Hoepli.) Fr. 3.50

NEILSON (R. M.). The steam turbine. 2d edition, revised and enlarged. New York, Longmans, 1903. 8vo. 17 + 294 pp. Cloth. $3.60

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SCHLITT (R.). Untersuchungen über einen Specialfall des Problems der drei Körper mit nahezu periodischer Lösung. (Diss.) Kiel, 1903. 8vo. 30 pp. 3 plates.

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TETMAJER (L. VON). Die angewandte Elastizitäts- und Festigkeitslehre, auf Grund der Erfahrung bearbeitet. 2te, vollständig umgearbeitete Auflage. Wien, Deuticke, 1904. 8vo. 15 + 565 pp., 10 plates. M. 16.00

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WALKER (T.). Introduction to physical chemistry. 3d edition. New York, Macmillan, 1903. 8vo. 12 + 368 pp. Cloth. $3.25

WALLIS (F.). See HAWKINS (C. C.).

WERNICKE (AD.). Lehrbuch der Mechanik in elementarer Darstellung mit Anwendungen und Uebungen aus den Gebieten der Physik und Technik. Teil I. 3te (Schluss-) Abteilung. 4te, völlig umgearbeitete Auflage, von Al. Wernicke. Braunschweig, Vieweg, 1903. 8vo. 11 pp. and pp. 811–1635.

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